数理解析研究所講究録
第 1826 巻 2013 年 28-35

On the image of the Saito-Kurokawa lifting over a totally real number field and the Maass relation

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1 Introduction

H. Saito and N. Kurokawa independently conjectured that there exists a lifting from an eigenform $\varphi\in S_{2k}(\mathrm{SL}_2(\mathbb{Z}))$ with k odd to an eigenform $\Phi(Z)=\sum_{B}A(B)\exp(2\pi\sqrt{-1}\,\mathrm{tr}(BZ))\in S_{k+1}(\mathrm{Sp}_2(\mathbb{Z}))$ such that

$$L(s, \Phi, \mathrm{sp})=\zeta(s-k)\zeta(s-k+1)L(s, \varphi).$$

Here $\zeta(s)$ is the Riemann zeta function, and $L(s, \Phi, \mathrm{sp})$ is the spin L-function of Φ. Moreover they conjectured that the Fourier coefficient $A(B)$ satisfies the Maass relation

$$A(B)=\sum_{d|\varepsilon(B)}d^{k}A(B_{d})$$

for any nonzero matrix $B\in S_{2}^{*}(\mathbb{Z})$, where $S_{2}^{*}(\mathbb{Z})$ is a set of all half integral symmetric matrix of size 2×2. Here the summation runs over all positive integer d which divide $\varepsilon(B)=\gcd(b_{11}, 2b_{12}, b_{22})$ for $B=(b_{ij})\in S_{2}^{*}(\mathbb{Z})$, and B_{d} is defined by $B_{d}=\begin{pmatrix}1 & b_{12}/d\\ b_{12}/d & b_{11}b_{22}/d^{2}\end{pmatrix}$. The conjecture was proved by Maass, Andrianov and Zagier (see [2], [17]). Then the lifting is called the Saito-Kurokawa lifting.

Naturally, we can consider the generalization of the Saito-Kurokawa lifting, i.e. we consider the lifting from a Hilbert modular form to a Hilbert-Siegel modular form over a totally real number field. In fact, Piatetski-Shapiro [12] and Schmidt [13] proved the existence of the generalized Saito-Kurokawa lifting using representation theory.

The main purpose of this paper is to give a Fourier coefficient formula of the lifted form and a generalization of the Maass relation.

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The auther would like to express his gratitude to Professor Tamotsu Ikeda for his encouragement and valuable advice. He also thanks Professor H. Kojima for his comment about [8].

2 Hilbert-Siegel modular form

In this section, K is a totally real number field of degree $d = [K : \mathbb{Q}]$, $\mathcal{O}_K$ is the ring of integers of K, and $\mathcal{D}_K$ is the different of K relative to $\mathbb{Q}$. Let $\mathfrak{H}_n$ be the Siegel upper half space of degree n, i.e.,

$$\mathfrak{H}_n = \{ Z \in M_n(\mathbb{C}) \mid {}^tZ = Z,\ \operatorname{Im} Z > 0 \}.$$

Let $\mathrm{GSp}_n^+(\mathbb{R}) = \{ g \in \mathrm{GL}_{2n}(\mathbb{R}) \mid {}^tgJ_ng = \nu(g)J_n,\ \nu(g) > 0 \}$, where $J_n = \left(\begin{smallmatrix} 0 & -1_n \\ 1_n & 0 \end{smallmatrix}\right)$. For $Z = (Z_1, \dots, Z_d) \in \mathfrak{H}_n^d$ and $M = (M_1, \dots, M_d) \in \mathrm{GSp}_n^+(\mathbb{R})^d$ with $M_i = \left(\begin{smallmatrix} A_i & B_i \\ C_i & D_i \end{smallmatrix}\right)$, put

$$MZ = (M_1Z_1, \dots, M_dZ_d) \in \mathfrak{H}_n^d,$$

where $M_iZ_i = (A_iZ_i + B_i)(C_iZ_i + D_i)^{-1}$. Let $x = (x_1, \dots, x_d) \in \mathbb{R}_{\geq 0}^d$ (resp. $x \in \mathbb{C}^d$) and $\kappa = (\kappa_1, \dots, \kappa_d) \in \mathbb{R}^d$ (resp. $\kappa \in \mathbb{Z}^d$). We define a multi-index notation x^κ by

$$x^\kappa = \prod_{i=1}^d x_i^{\kappa_i}.$$

Put

$$j(M, Z)^\kappa = \det(M)^{-\kappa/2} \det(CZ + D)^\kappa,$$

using the multi-index notation. Here we use following abbreviations:

$$\det(M) = (\det M_1, \dots, \det M_d),$$
$$CZ + D = (C_1Z_1 + D_1, \dots, C_dZ_d + D_d),$$

for $Z = (Z_1, \dots, Z_d) \in \mathfrak{H}_n^d$ and $M = (M_1, \dots, M_d) \in \mathrm{GSp}_n^+(\mathbb{R})^d$ with $M_i = \left(\begin{smallmatrix} A_i & B_i \\ C_i & D_i \end{smallmatrix}\right)$. Fix the mutually different real embeddings $K \ni x \mapsto x^{(i)} \in \mathbb{R}\,(i = 1, \dots, d)$ with $x^{(1)} = x$. Let K_∞ and $\mathrm{GSp}_n^+(K_\infty)$ be the archimedian parts of $\mathbb{A}_K$ and $\mathrm{GSp}_n^+(\mathbb{A}_K)$. Then we identify $\mathbb{R}^d$ (resp. $\mathrm{GSp}_n^+(\mathbb{R})^d$) with K_∞ (resp. $\mathrm{GSp}_n^+(K_\infty)$) by $K \ni x \mapsto (x^{(1)}, \dots, x^{(d)}) \in \mathbb{R}^d$. Let Φ be a function on $\mathfrak{H}_n^d$, and $M \in \mathrm{GSp}_n^+(K) \subset \mathrm{GSp}_n^+(\mathbb{R})^d$. Define a function $\Phi|_\kappa M$ by

$$(\Phi|_\kappa M)(Z) = j(M, Z)^{-\kappa}\Phi(MZ).$$

Put $W = \mathrm{GSp}_n^+(K_\infty) \times \prod_{v<\infty} W_v$, where $W_v = \mathrm{GSp}_n(\mathcal{O}_{K_v})$. Take h elements $t_1, \dots, t_h$ of $\mathbb{A}_K^\times$ so that $t_{\lambda,v} = 1$ for all $v|\infty$ and $t_1\mathcal{O}_K, \dots, t_h\mathcal{O}_K$ form a complete set of representatives for narrow ideal classes of K. Here we denote by $y\mathcal{O}_K$ the fractional ideal of K associated to $y \in \mathbb{A}_K^\times$. Let $x_\lambda = \mathrm{diag}(t_\lambda^{-1} I_n, I_n)$. Then we have $\mathrm{GSp}_n(\mathbb{A}_K) = \bigsqcup_{\lambda=1}^{h} \mathrm{GSp}_n(K) x_\lambda W$, where $\bigsqcup$ is the disjoint union. Put $\Gamma_\lambda = x_\lambda W x_\lambda^{-1} \cap \mathrm{GSp}_n(K)$.

Now we assume $K \neq \mathbb{Q}$ or $n \neq 1$. Then a Hilbert-Siegel modular form of weight $\kappa \in \mathbb{Z}^d$ with respect to Γ_λ is a holomorphic function $\Phi : \mathfrak{H}_n^d \to \mathbb{C}$ such that $\Phi|_\kappa M = \Phi$ for any $M \in \Gamma_\lambda$. We denote the space of Hilbert-Siegel modular forms of weight κ and degree n with respect to Γ_λ by $M_\kappa^{(n)}(\Gamma_\lambda)$. Let $\Phi \in M_\kappa^{(n)}(\Gamma_\lambda)$ and $M \in \mathrm{GSp}_n^+(K)$. It is known that $\Phi|_\kappa M$ has a Fourier expansion

$$(\Phi|_\kappa M)(Z) = \sum_B A_M(B) \exp(2\pi\sqrt{-1}\,\mathrm{Tr}(BZ)).$$

Here $\mathrm{Tr}(BZ) = \mathrm{tr}(\sum_{i=1}^d B^{(i)} Z_i)$, and the summation runs over $B \in \mathcal{S}_n^*(t_\lambda \mathcal{D}_K^{-1})$ such that $B^{(i)}$ $(i = 1, \ldots, d)$ are all positive semi-definite. We define a space of cusp forms $S_\kappa^{(n)}(\Gamma_\lambda)$ by

$$S_\kappa^{(n)}(\Gamma_\lambda) = \left\{ \Phi \in M_\kappa^{(n)}(\Gamma_\lambda) \;\middle|\; \begin{array}{c} A_M(B) = 0 \text{ unless } B \gg 0 \\ \text{for any } M \in \mathrm{GSp}_n^+(K) \end{array} \right\}.$$

Here $B \gg 0$ if $B^{(i)} > 0$ for $i = 1, \ldots, d$. Put $M_\kappa^{(n)} = \prod_{\lambda=1}^h M_\kappa^{(n)}(\Gamma_\lambda)$ and $S_\kappa^{(n)} = \prod_{\lambda=1}^h S_\kappa^{(n)}(\Gamma_\lambda)$. Let $(\Phi_1, \ldots, \Phi_h) \in M_\kappa^{(n)}$. We define a function $\tilde{\Phi}$ on $\mathrm{GSp}_n(\mathbb{A})$ by $\tilde{\Phi}(\alpha x_\lambda w) = \Phi_\lambda|_\kappa w_\infty(\boldsymbol{i})$, where $\alpha \in \mathrm{GSp}_n(K)$, $w \in W$, and $\boldsymbol{i} = (\sqrt{-1}\, I_n, \ldots, \sqrt{-1}\, I_n) \in \mathfrak{H}_n^d$. We identify $\tilde{\Phi}$ with $(\Phi_1, \ldots, \Phi_h) \in M_\kappa^{(n)}$.

3 Siegel series

In this section K, $\mathcal{O}$, $\mathfrak{p}$ and ϖ denote a finite algebraic extension of $\mathbb{Q}_p$, the integral closure of $\mathbb{Z}_p$ in K, maximal ideal of $\mathcal{O}$, and a prime element of K, respectively. Let $q = [\mathcal{O} : \mathfrak{p}]$, and $|\cdot|$ be the normalized absolute value on K, i.e. $|\varpi| = q^{-1}$. Let R be a fractional ideal of K. Let $\mathcal{S}_n(R)$ denote the set of symmetric matrices of size n with entries in R, and put $\mathcal{S}_n^*(R) = \{ (b_{ij}) \in \mathcal{S}_n(K) \mid b_{ii}, 2b_{ij} \in R, (i,j = 1, \ldots, n) \}$. Let $\chi(x) = \exp(-2\pi\sqrt{-1}y)$ be a character of K with $y \in \mathbb{Z}[1/p]$ such that $\mathrm{tr}_{K/\mathbb{Q}_p}(x) - y \in \mathbb{Z}_p$. Then we have $\{ B \in \mathcal{S}_n(K) \mid \chi(\mathrm{tr}(B\mathcal{S}_n(\mathcal{O}))) = 1 \} = \mathcal{S}_n^*(\mathcal{D}_K^{-1})$. Here $\mathcal{D}_K$ is the different of K relative to $\mathbb{Q}_p$. For $S \in \mathcal{S}_n(K)$, we put $\nu(S) = [S\mathcal{O}^n + \mathcal{O}^n : \mathcal{O}^n]$.

Given $B \in \mathcal{S}_n^*(\mathcal{D}_K^{-1})$, we define a formal Dirichlet series $b(B, s)$ by

$$b(B,s) = \sum_{R \in \mathcal{S}_n(K)/\mathcal{S}_n(\mathcal{O})} \chi(\mathrm{tr}(BR))\nu(R)^{-s}.$$

We call the series $b(B, s)$ a Siegel series of degree n over K. Here $\nu(R)$ and $\chi(\mathrm{tr}(BR))$ depend only on the class R and then the sum is formaly well-defined. It is known that $b(B, s)$ is convergent if $\mathrm{Re}(s)$ sufficiently large for any given $B \in \mathcal{S}_n^*(\mathcal{D}_K^{-1})$.

Now we consider the Siegel series in the case $n = 2$. We fix $F = \mathrm{diag}(\varpi^{l_1}, \varpi^{l_1+l_2}) \in \mathcal{F}$ and nonzero matrix $B \in \mathcal{S}_2^*(\mathcal{D}_K^{-1})$. Let $\varepsilon(B)$ be the minimal integral ideal $\mathfrak{a}$ satisfying $B \in \mathcal{S}_2^*(\mathcal{D}_K^{-1}\mathfrak{a})$, and $\alpha_1 = \mathrm{ord}_\mathfrak{p}\, \varepsilon(B)$. We put $\delta \in K$ so that $\delta\mathcal{O} = \mathcal{D}_K$. For $N \in K^\times$, we define $\alpha(N)$ and ξ_N by $\alpha(N) = \frac{1}{2}\big(\mathrm{ord}_\mathfrak{p}\, N - \mathrm{ord}_\mathfrak{P}\, \mathcal{D}_{K(\sqrt{N})/K}\big) + \mathrm{ord}_\mathfrak{p}\, \delta$ and

$$\xi_N = \begin{cases} 1 & \text{if } N \in K^{\times 2}, \\ -1 & \text{if } K(\sqrt{N})/K \text{ is unramified extension,} \\ 0 & \text{if } K(\sqrt{N})/K \text{ is ramified extension,} \end{cases}$$

respectively. Here $\mathcal{D}_{K(\sqrt{N})/K}$ is different of $K(\sqrt{N})$ relative to K, and $\mathfrak{P}$ is maximal ideal of $K(\sqrt{N})$. Put $t = -\det(2\delta B)$, $\alpha = \alpha(t) - \mathrm{ord}_\mathfrak{p}\, \delta$, and $\xi_B = \xi_t$.

Theorem 3.1 Let $B \in \mathcal{S}_2^*(\mathcal{D}_K^{-1})$, and $\det B \neq 0$, then

$$b(B,s) = \frac{(1-q^{-s})(1-q^{2-2s})}{1-\xi_B\, q^{1-s}} F(B, q^{-s})$$

where $F(B,X)$ is a polynomial of X with integral coefficients:

$$F(B,X) = \sum_{l=0}^{\alpha_1} (q^2X)^l \Big\{ \sum_{m=0}^{\alpha-l} (q^3X^2)^m - \xi_B\, qX \sum_{m=0}^{\alpha-l-1} (q^3X^2)^m \Big\}.$$

Corollary 3.2 Let $\widetilde{F}(B,X) = X^{-\alpha} F(B, q^{-3/2}X)$, then

$$\widetilde{F}(B,X) = \sum_{l=0}^{\alpha_1} q^{l/2} \Big(\frac{X^{\alpha-l+1} - X^{-\alpha+l-1}}{X - X^{-1}} - \xi_B\, q^{-1/2} \frac{X^{\alpha-l} - X^{-\alpha+l}}{X - X^{-1}} \Big).$$

4 Main results

Assume $(\phi_\lambda) \in S_{2\kappa}^{(1)}$ is a Hecke eigenform with the Satake parameter $\{\alpha_v, \alpha_v^{-1}\}$ for $v < \infty$. Let $(\varPhi_\lambda) \in S_{\kappa+1}^{(2)}$ be the image of the Saito-Kurokawa lifting of (ϕ_λ), and $A_\lambda(B)$ the B^{th} Fourier coefficient of

$$\varPhi_\lambda = \sum_B A_\lambda(B) \exp(2\pi\sqrt{-1}\,\mathrm{Tr}(BZ)) \in S_{\kappa+1}^{(2)}(\varGamma_\lambda).$$

Here $\mathrm{Tr}(BZ) = \mathrm{tr}(\sum_{i=1}^d B^{(i)} Z_i)$, and the summation runs over $B \in \mathcal{S}_2^*(t_\lambda \mathcal{D}_K^{-1})$ such that $B^{(i)}$ $(i = 1, \ldots, d)$ are all positive semi-definite. Then the first main result is as follows:

Theorem 4.1 (Fourier coefficient formula) The following assertion holds:

$$A_\lambda(B) = C_B N(t_\lambda \mathcal{O}_K)^{3/2} \det B^{\kappa/2-1/4} \prod_{v<\infty} \widetilde{F}_v(t_{\lambda,v}^{-1} B, \alpha_v)$$

using the multi-index notation. Here $\widetilde{F}_v(B,X)$ is the Laurent polynomial in Corollary 3.2, and $t_\lambda = (t_{\lambda,v}) \in \mathbb{A}_K^\times$ are as in §2.

Moreover the constant C_B satisfies $C_{rB[A]} = \operatorname{sgn}(\det A)^{\kappa+1} C_B$ for any $A \in \mathrm{GL}_2(K)$, $r \in K_+^\times$. Thus the constant C_B depends only on $\det B \mod (K_+^\times)^2$.

For given $B \in \mathcal{S}_2^*(t_\lambda \mathcal{D}_K^{-1})$ and $\delta_v \in K_v$ such that $\delta_v \mathcal{O}_{K_v} = \mathcal{D}_{K_v}$, let $t_B = -\det(2B)$, and $\alpha_{\lambda,v}(B) = \frac{1}{2}\big(\operatorname{ord} t_B - \operatorname{ord} \mathcal{D}_{K_v(\sqrt{t_B})/K_v}\big) + \operatorname{ord}_v \delta_v - \operatorname{ord}_v t_{\lambda,v}$. Let $\varepsilon_\lambda(B)$ be the minimal integral ideal $\mathfrak{a}$ satisfying $B \in \mathcal{S}_2^*(t_\lambda \mathcal{D}_K^{-1}\mathfrak{a})$, Put $\mathfrak{f}_{\lambda,B} = \prod_{v<\infty} \mathfrak{p}_v^{\alpha_{\lambda,v}(B)}$, where $\mathfrak{p}_v$ is the maximal ideal of K_v. For an integral ideal $\mathfrak{a}|\varepsilon_\lambda(B)$, we take a fractional ideal $t_\mu \mathcal{O}_K$ and $\eta \in K_+^\times = \{\, x \in K^\times \mid x \gg 0 \,\}$ so that $t_\lambda \mathfrak{a} = t_\mu(\eta)$. We put $A_\lambda^0(B/\mathfrak{a}) = A_\mu^0(\eta^{-1}B)$. Then $A_\lambda^0(B/\mathfrak{a})$ is independent of the choice of η. Let $\mu(\mathfrak{a})$ be the Möbius function, and $S_{\kappa+1}^{(2),\,\mathrm{SK}}$ the subspace of $S_{\kappa+1}^{(2)}$ spanned by the Saito-Kurokawa lifting of all Hecke eigenforms in $S_{2\kappa}^{(1)}$. Then the second main result is as follows:

Theorem 4.2 (Maass relation) Let $\Phi = (\Phi_\lambda) \in S_{\kappa+1}^{(2),\,\mathrm{SK}}$, and $A_\lambda(B)$ the B^{th} Fourier coefficient of Φ_λ. Put $A_\lambda^0(B) = N(t_\lambda \mathcal{O}_K)^{-3/2} \det B^{-\kappa/2+1/4} A_\lambda(B)$. Then for $N \in K^\times/K^{\times 2}$ and an integral ideal $\mathfrak{a}$, there exists a $\mathbb{C}$-valued function $T_\Phi(N,\mathfrak{a})$ such that

$$A_\lambda^0(B) = \sum_{\mathfrak{a}|\varepsilon_\lambda(B)} N(\mathfrak{a})^{1/2} T_\Phi(t_B, \mathfrak{f}_{\lambda,B}\mathfrak{a}^{-1}),$$

for any $B \in \mathcal{S}_2^*(t_\lambda \mathcal{D}_K^{-1})$ and λ. Here $\mathfrak{a}$ runs over all integral ideals dividing $\varepsilon_\lambda(B)$.

Next we assume that the narrow class number of K is one. Take $0 \ll \delta \in \mathcal{O}_K$ so that $\mathcal{D}_K = \delta \mathcal{O}_K$. For $0 \ll B = \delta^{-1}(b_{ij}) \in \mathcal{S}_2^*(\mathcal{D}_K^{-1})$ and $0 \ll d \in \mathcal{O}_K$, put $B_d = \delta^{-1}\begin{pmatrix} 1 & b_{12}/d \\ b_{12}/d & b_{11}b_{22}/d^2 \end{pmatrix}$. Let $\Phi = \Phi_1 \in S_{\kappa+1}^{(2),\,\mathrm{SK}}$, $A(B) = A_1(B)$, $A^0(B) = A_1^0(B)$ and $t_1\mathcal{O}_K = \mathcal{O}_K$. Then we can take $T_\Phi(t_B, \mathfrak{f}_{1,B}(d)^{-1}) = A^0(B_d)$. Thus we have the same formulation as the classical Maass relation.

Corollary 4.3 Assume above setting. Then the Fourier coefficients satisfy a linear relation

$$A(B) = \sum_{d\mathcal{O}_K|\varepsilon(B)} d^\kappa A(B_d)$$

for any $B \in \mathcal{S}_2^*(\mathcal{D}_K^{-1})$ using the multi-index notation. Here $0 \ll d \in \mathcal{O}_K$, and for given B, $d^\kappa A(B_d)$ depends only on the ideal $d\mathcal{O}_K$.

5 An example

In this section, we give an example of the Saito-Kurokawa lifting, and we see the lifted form satisfies the Maass relation. Let $K = \mathbb{Q}(\sqrt{5})$, $Z = (Z_1, Z_2) \in \mathfrak{H}_2^2$, $\varepsilon = \frac{1+\sqrt{5}}{2}$ and $Q \in M_8(\mathcal{O}_K)$ which is positive definite and even-integral. Then we define a theta function $\Theta_Q(Z)$ by

$$\Theta_Q(Z) = \sum_{X \in M_{8,2}(\mathcal{O}_K)} \exp(\pi\sqrt{-1}\sigma\Big(\frac{Q[X]Z}{\varepsilon\sqrt{5}}\Big)).$$

Here $\sigma\Big(\frac{Q[X]Z}{\varepsilon\sqrt{5}}\Big) = \mathrm{tr}(\Big(\frac{Q[X]}{\varepsilon\sqrt{5}}\Big)^{(1)} Z_1) + \mathrm{tr}(\Big(\frac{Q[X]}{\varepsilon\sqrt{5}}\Big)^{(2)} Z_2)$, and $x^{(i)}$ is a real embeddings $K \ni x \to x^{(i)} \in \mathbb{R}$ $(i = 1, 2)$. Note that the narrow class number of $K = \mathbb{Q}(\sqrt{5})$ is one, and the different $\mathcal{D}_K$ is generated by a totally positive element $\varepsilon\sqrt{5}$ of K. Then $\Theta_Q(Z)$ is a Hilbert-Siegel modular form of weight $\kappa = (4, 4)$ for $\mathrm{Sp}_2(\mathcal{O}_K)$, and the Fourier expansion is

$$\begin{aligned}\Theta_Q(Z) &= \sum_{X \in M_{8,2}(\mathcal{O}_K)} \exp(\pi\sqrt{-1}\sigma\Big(\frac{Q[X]Z}{\varepsilon\sqrt{5}}\Big)), \\ &= \sum_{B \in \mathcal{S}_2^*(\mathcal{D}_K^{-1})} A(Q, B) \exp(2\pi\sqrt{-1}\sigma(BZ)).\end{aligned}$$

Here

$$A(Q, B) = \sharp\Big\{ X \in M_{8,2}(\mathcal{O}_K) \Big| Q[X] = 2\varepsilon\sqrt{5}B \Big\}.$$

Thus we can compute the B^{th} Fourier coefficient of Θ_Q counting the solutions of $Q[X] = 2\varepsilon\sqrt{5}B$.

By Maass [10], there exist exactly two inequivalent classes of even quadratic form with determinant one and eight variables over $\mathbb{Q}(\sqrt{5})$. These two classes are represented by following matrices: $F_4^2 = F_4 \oplus F_4$ and E_8.

Here $F_4 = \begin{pmatrix} 2 & -1 & 0 & 1-\varepsilon \\ -1 & 2 & -1 & \varepsilon - 1 \\ 0 & -1 & 2 & \varepsilon \\ 1-\varepsilon & \varepsilon - 1 & \varepsilon & 2 \end{pmatrix}$, and $E_8 \in \mathrm{GL}_8(\mathbb{Z})$ is a positive definite even unimodular matrix over $\mathbb{Z}$.

Put

$$s_4(Z) = \frac{1}{2880}(\Theta_{F_4^2}(Z) - \Theta_{E_8}(Z)).$$

Proposition 5.1 $s_4(Z) \in S_{(4,4)}(\mathrm{Sp}_2(\mathcal{O}_K))$.

Let $S^{\mathrm{SK}}_{\kappa+1}(\mathrm{Sp}_2(\mathcal{O}_K))$ be the subspace of $S_{\kappa+1}(\mathrm{Sp}_2(\mathcal{O}_K))$ spanned by the Saito-Kurokawa lifting of all Hecke eigenforms in $S_{2\kappa}(\mathrm{SL}_2(\mathcal{O}_K))$.

Proposition 5.2 The following assertion holds:

1. $\dim S^{\mathrm{SK}}_{(4,4)}(\mathrm{Sp}_2(\mathcal{O}_K)) = \dim S_{(4,4)}(\mathrm{Sp}_2(\mathcal{O}_K)) = 1$.
2. $s_4(Z) \in S^{\mathrm{SK}}_{(4,4)}(\mathrm{Sp}_2(\mathcal{O}_K))$.

We show some Fourier coefficients $A(F_4^2, B)$, $A(E_8, B)$ and $A(B)$ of $\Theta_{F_4^2}$, Θ_{E_8} and s_4, respectively:

N	(a,b,c)	$A(F_4^2,B)$	$A(E_8,B)$	$A(B)$
5	$(2,\varepsilon,2)$	2880	0	1
80	$(2,2\varepsilon,8)$	1918080	1814400	36
80	$(4,2\varepsilon,4)$	2102400	1814400	100
9	$(2,1,2)$	4800	13440	-3
144	$(2,2,8)$	8083200	8117760	-12
144	$(4,2,4)$	8390400	8977920	-204

Here $N = N_{K/\mathbb{Q}}(\det(2\varepsilon\sqrt{5}B))$, and (a,b,c) is an abbreviation for B so that $2\varepsilon\sqrt{5}B = \left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)$.

Put $(a,b,c) = B$, $A(a,b,c) = A(B)$ using above abbreviation. Then we see the Maass relation (Corollary 4.3) holds:

$$A(4,2\varepsilon,4) = 100 = A(2,2\varepsilon,8) + 64\,A(2,\varepsilon,2),$$

$$A(4,2,4) = -204 = A(2,2,8) + 64\,A(2,1,2).$$

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